Homological algebra exercise sheet Week 14

1. Suppose that $F : \mathbf{K}^+(\mathcal{A}) \to \mathbf{K}(\mathcal{C})$ is a morphism of triangulated categories and $\mathcal{B}$ is a Serre subcategory of $\mathcal{A}$ (see Exercise sheet 12 for definition and properties). If $\mathcal{A}$ has enough injectives, show that the restriction of $\mathbf{R}^+F$ to $\mathbf{D}^+_{\mathcal{B}}\mathcal{A}$ is the derived functor $\mathbf{R}^+_{\mathcal{B}}F$.

2. In general we cannot get a unbounded derived functor by taking unbounded projective (injective) resolutions. Here is a counterexample.

   (a) Show that for the $R = \mathbb{Z}/4\mathbb{Z}$-module category, the following complex consists of projective objects and is quasi-isomorphic to the zero complex.
   $$P^* : \cdots \xrightarrow{2} R \xrightarrow{2} R \xrightarrow{2} R \to \cdots$$

   (b) Show that $P^* \otimes_R \mathbb{Z}/2\mathbb{Z}$ is not quasi-isomorphic to zero complex.

   This shows that applying an exact functor on quasi-isomorphic unbounded complexes of projective objects can give different result in derived category. Remember that in the bounded case, the result is determined by Comparison Theorem.

3. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor of abelian categories and suppose $\mathcal{A}$ has enough injectives (so the usual derived functors exist). We let the cohomological dimension of F be the first n such that $R^iF = 0\ \forall i > n$. We make a similar definition when $\mathcal{A}$ has enough projectives for the homological dimension.

   (a) Show if F has finite cohomological dimension, then every exact complex of F-acyclic objects is an F-acyclic complex.

   (b) Show if F has finite cohomological dimension, $\mathbf{R}F$ exists on $D(\mathcal{A})$ (Hint: Consider K' the full subcategory of $K(\mathcal{A})$ consisting of complexes of F-acyclic objects in $\mathcal{A}$, and use the Generalized Existence Theorem).

4. Consider the derived functor $\mathbf{L}\,\mathrm{Tot}^{\oplus}(\bullet \otimes B)$ from $D(R\text{-}\mathbf{mod})$ to $D(\mathbf{Ab})$, where R is a commutative ring and B a cochain complexe of R modules. Show that $A \otimes^L B$ is naturally isomorphic to $\mathbf{L}\,\mathrm{Tot}^{\oplus}(\bullet \otimes B)A$ in $D(\mathbf{Ab})$.

5. Show that the total tensor product may be refined to a functor
$$\otimes^L_R : D^-(R_1\text{-}\mathbf{mod}\text{-}R) \times D^-(R\text{-}\mathbf{mod}\text{-}R_2) \to D^-(R_1\text{-}\mathbf{mod}\text{-}R_2)$$

in the sense that the diagram

$$\begin{array}{ccc} D^-(R_1\textbf{-mod-}R) \times D^-(R\textbf{-mod-}R_2) & \xrightarrow{\otimes^L_R} & D^-(R_1\textbf{-mod-}R_2) \\ \downarrow & & \downarrow \\ D^-(\textbf{mod-}R) \times D^-(R\textbf{-mod}) & \xrightarrow{\otimes^L_R} & D^-(\textbf{Ab}) \end{array}$$

commutes where the vertical arrows are the forgetful functors. For R a commutative ring, refine it further to

$$\otimes^L_R : D^-(R\textbf{-mod}) \times D^-(R\textbf{-mod}) \to D^-(R\textbf{-mod})$$

and show there is a natural isomorphism $A \otimes^L_R B \cong B \otimes^L_R A$.

6. If $F : K'' \to K(\mathcal{D}), G : K' \to K'', H : K \to K'$ are three consecutive morphisms of triangulated categories, where $K \subset K(\mathcal{A}), K' \subset K(\mathcal{B}), K'' \subset K(\mathcal{C})$ are localizing triangulated subcategories. Assume all necessary derived functors exist and can be composed. Using the notation of the composition theorem, show that as natural transformations from $\mathbf{R}(FGH)$ to $\mathbf{R}F \circ \mathbf{R}G \circ \mathbf{R}H$ we have $\zeta_{G,H} \circ \zeta_{F,GH} = \zeta_{F,G} \circ \zeta_{FG,H}$.